Power Series Solutions of second order linear ODE.

The method is easily generalized to higher linear ODE.

Recall: A power series about a point x_0 is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$$

Abel Theorem: For every power series there is a number R, radius of convergence, such that

(i) If $|x-x_0| < R$, then the series converges

(ii) If $|x-x_0| > R$, then series diverges

Rmk: When $|x-x_0| = R$, we can't tell if the series converges in general. The convergence in this case must be analyzed case-by-case.

Rmk: $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ or $R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{a_n}}$

by ratio test by root test.

Why study power series?

(i) Every elementary function (power, exp., log., trig., inverse trig.) admits a power series expansion about an ordinary point., i.e., where the function is analytic, meaning where the function is infinitely differentiable, plus some other conditions we don't know about. This is done by Taylor series:

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2!}f''(x_0)(x-x_0)^2 + \cdots + \frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n + \cdots$$

Example: $e^x = 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + \cdots, \quad x_0 = 0.$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots, \quad x_0 = 0.$$

The series recovers the function as long as the series converges.

(ii) There are many function that cannot be expressed in terms of elem. funcs., yet admits a power series expression.

Example: $\int_0^x e^{-x^2}dx = \int_0^x \left[1 + (-x^2) + \frac{1}{2!}(-x^2)^2 + \cdots + \frac{1}{n!}(-x^2)^n + \cdots\right]dx$

$$= \int_0^x \sum_{n=0}^{\infty} \frac{1}{n!}(-1)^n x^{2n} = \sum_{n=0}^{\infty} \frac{1}{n!}(-1)^n \frac{1}{2n+1} x^{2n+1}$$

$$= x - \frac{1}{3}x^3 + \frac{1}{2!\cdot 5}x^5 + \cdots + \frac{(-1)^n}{n!\cdot(2n+1)}x^{2n+1} + \cdots$$

This power series cannot be expressed in terms of elem. funcs.

For many generic linear ODEs

$$\mathcal{L}y = y'' + p(t)y' + q(t)y = 0$$

the solutions cannot be described by elem. funcs. Yet power series works.

$$\mathcal{L} = \left[\left(\frac{d}{dt}\right)^2 + p(t)\cdot\frac{d}{dt} + q(t)\cdot\right]$$

$\boxed{\text{Dr. Gross's video lecture.}}$

Summary: $y'' + y = 0$

Setup — Set $y = \sum_{n=0}^{\infty} a_n x^n$

Compute LHS. Write it as a single power. $\quad y'' + y = \sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} + a_n\right]x^n = 0$

most difficult: shift exponent, unify sum indices

Recurrence $\quad a_{n+2} = \frac{-a_n}{(n+2)(n+1)}$

Solving the recurrence rel'n

$$\overset{n=0}{a_2 = \frac{a_0}{-2\cdot 1}}, \quad \overset{n=1}{a_3 = -\frac{a_1}{3\cdot 2}}, \quad \overset{n=2}{a_4 = -\frac{a_2}{4\cdot 3} = \frac{a_0}{4!}}, \quad \overset{n=3}{a_5 = \frac{-a_3}{5\cdot 4} = \frac{a_1}{5!}}$$

$$\underset{n=4}{a_6 = \frac{-a_4}{6\cdot 5} = -\frac{a_0}{6!}}, \quad \underset{n=5}{a_7 = \frac{-a_5}{7\cdot 6} = -\frac{a_1}{7!}}, \quad \ldots\ldots$$

Get the sol'n $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + \cdots$

$$= a_0 + a_1 x - \frac{a_0}{2!}x^2 - \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 - \frac{a_1}{5!}x^5 + \frac{a_0}{6!}x^6 + \frac{a_1}{7!}x^7 + \cdots$$

$$= a_0\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right)$$

$$+ a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{1}{7!}x^7 + \cdots\right)$$

4 Steps: (i) Set up the template: $y = \sum_{n=0}^{\infty} a_n x^n$

(ii) Compute LHS and express it as a single power series.

(iii) Get the recurrence relation and solve it with two arbitrary constants.

(iv) Formulate the sol'n

Example: $y'' - xy' + y = 0$. Find series sol'n about $x_0 = 0$

Step 1: $y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

Step 2: $y'' - xy' + y = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \underline{x \cdot \sum_{n=1}^{\infty} n a_n x^{n-1}} + \sum_{n=0}^{\infty} a_n x^n$

(multiplication)

$$= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n$$

$m=n-2$, $n=m+2$ (x^m) ; $m=n$, $n=m$ (x^m) ; $m=n$, $n=m$ (x^m)

(Unify the exponents to m.)

$$= \sum_{m+2=2}^{\infty} (m+2)(m+2-1)a_{m+2}x^m - \sum_{m=1}^{\infty} m a_m x^m + \sum_{m=0}^{\infty} a_m x^m$$

$$= \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}x^m - \sum_{m=1}^{\infty} m a_m x^m + \sum_{m=0}^{\infty} a_m x^m$$

(Unify sum indices according to the longest one)

$$= (0+2)(0+1)a_{0+2}x^0 + \sum_{m=1}^{\infty}(m+2)(m+1)a_{m+2}x^m \quad \text{Punch out 0th term}$$

$$- \sum_{m=1}^{\infty} m a_m x^m \quad \text{Keep it}$$

$$a_0 x^0 + \sum_{m=1}^{\infty} a_m x^m \quad \text{Punch out 0th term}$$

$$= 2a_2 + a_0 + \sum_{m=1}^{\infty}\left[(m+2)(m+1)a_{m+2} - m a_m + a_m\right]x^m = 0$$

Recall: $\sum_{m=0}^{\infty} c_m x^m = 0 \Rightarrow$ All $c_m = 0$.

Step 3: $\begin{cases} 2a_2 + a_0 = 0 \\ \text{For } m \geq 1, \ (m+2)(m+1)a_{m+2} - (m-1)a_m = 0 \end{cases}$

$$\Rightarrow \begin{cases} a_2 = -\frac{1}{2}a_0 \\ a_{m+2} = \frac{m-1}{(m+2)(m+1)}a_m. \quad m \geq 1 \end{cases}$$

Set a_0, a_1 arbitrary constants.

From first eqn: $a_2 = -\frac{1}{2}a_0$

From second eqn:

$m=1$ yields $a_3 = \frac{1-1}{(1+2)(1+1)} a_1 = 0$

$m=2$ $a_4 = \frac{2-1}{(2+2)(2+1)} a_2 = \frac{1}{4\cdot 3} a_2 = -\frac{1}{4!} a_0.$

$m=3$ $a_5 = \frac{3-1}{(3+2)(3+1)} a_3 = 0$

$m=4.$ $a_6 = \frac{4-1}{(4+2)(4+1)} = \frac{3}{6\cdot 5} a_4 = \frac{-3}{6!} a_0$

$m=5$ $a_7 = * a_5 = 0$

$m=6.$ $a_8 = \frac{6-1}{(6+2)(6+1)} a_6 = \frac{5}{8\cdot 7} a_6 = \frac{-5\cdot 3}{8!} a_0$

Step 4:

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 + \cdots$$

$$= a_0 + a_1 x - \frac{1}{2!} a_0 x^2 + 0 - \frac{1}{4!} a_0 x^4 + 0 - \frac{3}{6!} a_0 x^6 + 0 - \frac{5\cdot 3}{8!} a_0 x^8 + \cdots$$

$$= a_0 \left(1 - \frac{1}{2!} x^2 - \frac{1}{4!} x^4 - \frac{3}{6!} x^6 - \frac{15}{8!} x^8 + \cdots\right) + a_1 x.$$

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